Quantum Information Security

Andreas Winter
(ICREA & Universitat Autònoma de Barcelona)

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(kind of a tutorial)

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Outline

0. Quantum formalism

1. Motivation: quantum privacy amplification

2. Tools: (smoothed) min-entropy

3. Applications: QKD, quantum data hiding

0. Quantum formalism

We only need the "probabilistic core" of quantum theory, not the "mechanics".

[See (e.g.) M. M. Wilde, QIT, Cambridge University Press]

0. Quantum formalism

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Non-commutative probability theory:

- States (densities)
- Transformations (channels)
- Observable probabilities (measurements & Born rule)

[See (e.g.) M. M. Wilde, QIT, Cambridge University Press]

0. Quantum formalism

Each quantum system characterized by a complex Hilbert space $\mathcal{H}$, A, B, C, ...
Here: dimensions $|\mathcal{H}|$, $|A|$, ... $< \infty$.

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Dirac ("bra-ket") notation:

- Vectors $|\psi\rangle$
- Covectors $\langle\varphi|$
- Inner product $\langle\varphi|\psi\rangle$
- Outer product $|\psi\rangle\langle\varphi|$, in particular rank-one projector $|\psi\rangle\langle\psi|$

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- Vectors $|\psi\rangle$
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- Inner product $\langle\varphi|\psi\rangle$
- Outer product $|\psi\rangle\langle\varphi|$
- Hermitian conjugate $\dagger$:
  $|\psi\rangle^\dagger = \langle\psi|$, $\langle\varphi|^\dagger = |\varphi\rangle$
- Everything described in terms of operators $A \in L(\mathcal{H})$, $A^\dagger \in L(\mathcal{H})$.

[See (e.g.) M. M. Wilde, QIT, Cambridge University Press]

0-a. States

States on $\mathcal{H}$ are *density operators*:

$\rho \geq 0$ (*positive semidefinite*), $\mathrm{Tr}\ \rho = 1$.

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(Well, it's really matrices...)

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Spectral theorem: $\rho = \sum_j \lambda_j |e_j\rangle\langle e_j|$,
with ONB $|e_j\rangle$ and $\lambda_j \geq 0$ summing to 1.

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Extremal points: rank-one projectors
$|\psi\rangle\langle\psi|$ – *pure states*.

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With a distinguished ONB $|x\rangle$ ("computational basis"), have diagonal matrices $\rho = \sum_x p_x |x\rangle\langle x|$, isomorphic to the probability simplex.

[See (e.g.) M. M. Wilde, QIT, Cambridge University Press]

0-6. Measurement

Observations on quantum systems require an observable, which is a resolution of the identity (aka POVM):

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State as the "catalog of expectations" (Schrödinger)

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0-b. Measurement

Example: Discriminating equiprobable hypotheses ρ and σ. Need a binary POVM $(M, 1-M)$, with $0 \leq M \leq 1$.

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Want to maximize success probability

$$Pr\{succ\} = \frac{1}{2} Tr\, \rho M + \frac{1}{2} Tr\, \sigma(1-M)$$

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$$= \frac{1}{2} + \frac{1}{2} \operatorname{Tr} (\rho - \sigma) M$$

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$$= \frac{1}{2} + \frac{1}{4}\,\underbrace{\mathrm{Tr}\,(\rho-\sigma)(2M-1)}$$

$$\max = \|\rho-\sigma\|_1 \quad \text{(trace norm)}$$

$$= \mathrm{Tr}\,|\rho-\sigma|$$

[See (e.g.) M. M. Wilde, QIT, Cambridge University Press]

0-c. Composite systems

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The latter allows us to identify a POVM (M_y) on A with $(M_y \otimes 1)$ on $A \otimes B$. And for (N_z) on B, joint POVM $(M_y \otimes N_z)$.

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$S(A \otimes B)$ clearly contains all tensor product densities $\alpha \otimes \beta$ and their convex hull, but it has so much more...

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$S(A \otimes B)$ clearly contains all tensor product densities $\alpha \otimes \beta$ and their convex hull, but it has so much more...

Entanglement! For instance all pure states $|\psi\rangle\langle\psi|$ with $|\psi\rangle \neq |\alpha\rangle \otimes |\beta\rangle$.

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Composition of A and B (Hilbert spaces) is a system with tensor product Hilbert space $A \otimes B$.

Marginals: For a state $\rho^{AB} \in S(A \otimes B)$, the rule

$$\mathrm{Tr}\ \rho(X \otimes 1) = \mathrm{Tr}\ \rho^A X \qquad \text{(for all } X\text{)}$$

singles out uniquely the partial trace:

$$\rho^A = \mathrm{Tr}_B\, \rho^{AB}.$$

[See (e.g.) M. M. Wilde, QIT, Cambridge University Press]

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Purification: For every state $\rho \in S(A)$, there exists a unit vector $|\varphi\rangle \in A \otimes A'$, s.t. ρ is the marginal of the pure state:

$$\rho^A = Tr_{A'} |\varphi\rangle\langle\varphi|.$$

Any other pure state $|\psi\rangle\langle\psi|$ with this property is related to $|\varphi\rangle\langle\varphi|$ by a unitary conjugation on A':

$$|\psi\rangle = (1 \otimes U)|\varphi\rangle.$$

[See (e.g.) M. M. Wilde, QIT, Cambridge University Press]

0-d. Quantum channels

Like classical channels, quantum channels map densities to densities, i.e.

$$N: S(A) \longrightarrow S(B).$$

As it must respect convex combinations, extends uniquely a *linear, positive, trace preserving* map $N: L(A) \longrightarrow L(B)$.

However, want *complete positivity*, i.e.

$$N \otimes id_C : S(A \otimes C) \longrightarrow S(B \otimes C)$$

should be positive for all auxiliary C.

[See (e.g.) M. M. Wilde, QIT, Cambridge University Press]

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Quantum channel

$$\mathcal{N}: L(A) \longrightarrow L(B)$$

is a *linear, completely positive and trace preserving (cptp) map.*

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Basic example: unitary conjugation on A

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Another one: partial trace from AB to A

$$\mathcal{N}(\rho) = Tr_B \rho.$$

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0-d. Quantum channels

More examples:

1) Noiseless channel = identity id_A.

2) Constant channel $K(\rho) = \omega_0$.

3) Depolarizing channels

4) Amplitude damping channels

5) Phase damping channels

6) Erasure channel $\epsilon_q(\rho)=(1-q)\rho \oplus q|*\rangle\langle *|$

[See (e.g.) M. M. Wilde, QIT, Cambridge University Press]

0-d. Quantum channels

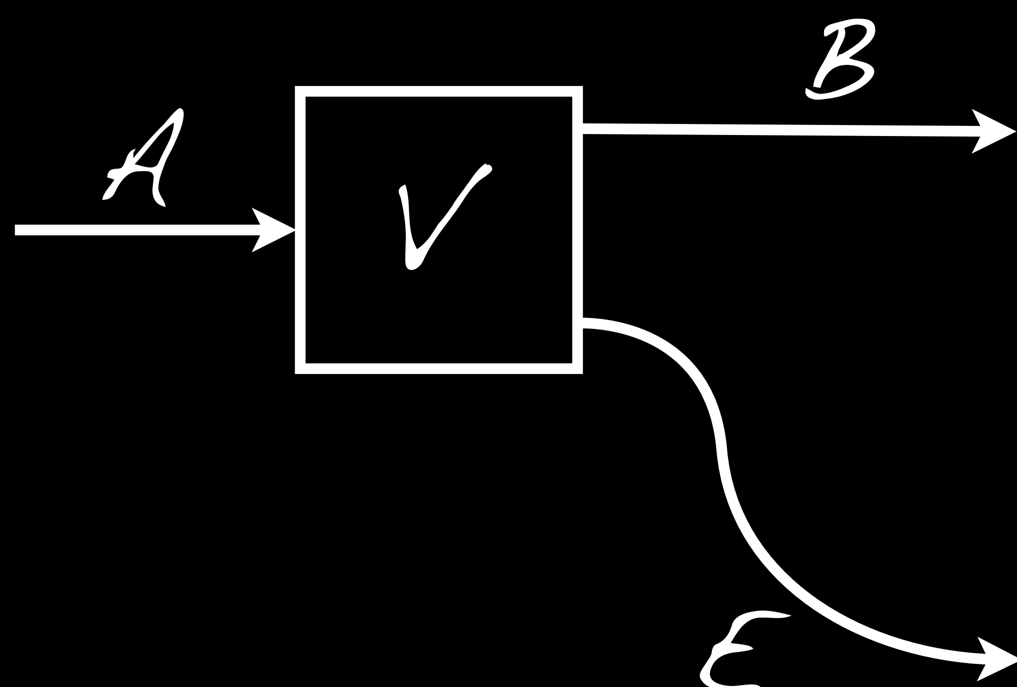
Stinespring: $\mathcal{N}(\rho) = \mathrm{Tr}_E\, V \rho V^\dagger$,

with an isometry $V : A \hookrightarrow B \otimes E$.

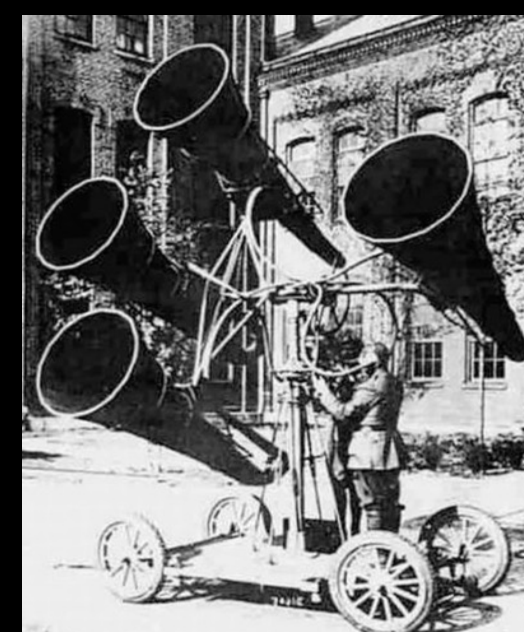
Complementary channel:

$$\widehat{\mathcal{N}}(\rho) = \mathrm{Tr}_B\, V \rho V^\dagger.$$

Alice



Bob

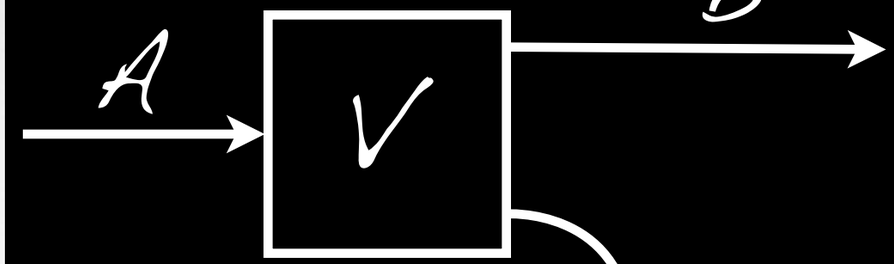


Eve

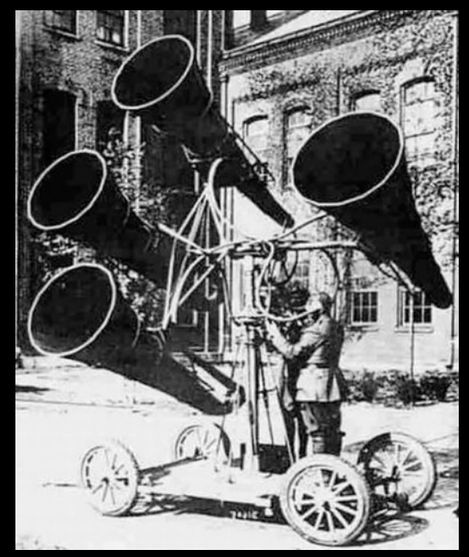
[M. M. Wilde, QIT, Cambridge University Press]

0-d. Quantum channels

Alice



Bob



Eve

It's a wiretap model! Note that automatically if Bob's channel is (nearly) noiseless, Eve's is (nearly) constant.

[Kretschmann et al., IEEE-IT 2008]

0-d. Quantum channels

Important property of cptp maps: trace norm is contractive. I.e., for any channel $\mathcal{N}$ and any operator ξ,

$$\|\mathcal{N}(\xi)\|_1 \leq \|\xi\|_1 .$$

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$$\|\mathcal{N}(\xi)\|_1 \leq \|\xi\|_1 .$$

Other metrics with this property, e.g. the *purified distance*

$$P(\rho,\sigma) = \min \tfrac{1}{2} \|\varphi-\psi\|_1 \text{ s.t. purifications } \varphi, \psi \text{ of } \rho, \sigma$$

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1. Privacy amplification

Alice has a "raw key" X (maybe shared with Bob), which is compromised: conditional on side information of Eve, the distribution of X is not uniform.

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But if we know $-\log \Pr\{X=x\} \geq h$, then 2-universal hash-function F to k bits yields Y=F(X) with

$$\| P_{YF} - U \otimes P_F \|_1 \leq 2^{-(h-k)/2},$$

where U is the uniform distribution.

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Including Eve explicitly, have Alice's X and Eve's Z jointly distributed. The optimal h is

$$H_{min}(X|Z) = -\log \max_{x,z} \Pr\{X=x|Z=z\}$$

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What can we do if Eve has quantum information? What can that conditional probability mean?

[Cf. R. Renner, PhD thesis, arXiv:quant-ph/0512258]

I. Privacy amplification

Now Alice and Eve have a joint state:

$$\rho^{XE} = \sum_x p_x |x\rangle\langle x|^X \otimes \rho_x^E.$$

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I. Privacy amplification

Now Alice and Eve have a joint state:

$$\rho^{XE} = \sum_x p_x |x\rangle\langle x|^X \otimes \rho_x^E.$$

It turns out the right definition is about comparing with operators $\lambda(1\otimes\sigma)$:

$$H_{min}(X|E) = -\log \min_{\lambda,\sigma} \lambda \text{ s.t. } \rho \le \lambda(1\otimes\sigma),$$

where σ is a state on E and λ is real.

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1. Privacy amplification

Let $H_{min}(X|E) = -\log \min_{\lambda,\sigma} \lambda$ s.t. $\rho \le \lambda(1 \otimes \sigma) \ge h$.

Then 2-universal hash-function F to k bits yields $Y=F(X)$ with

$$\| \rho^{YFE} - U \otimes \rho_F \otimes \rho^E \|_1 \le 2^{-(h-k)/2},$$

where U is the uniform distribution (maximally mixed state).

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...essentially optimal without knowing more about the joint state.

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2. Min-entropy calculus

Renner found that one could develop a meaningful one-shot information calculus based on conditional min-entropy:

$$H_{min}(A|B)_\rho = -\log \min_{\lambda,\sigma} \lambda \text{ s.t. } \rho^{AB} \leq \lambda(1^A \otimes \sigma^B),$$

for an arbitrary state ρ on $A \otimes B$.

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for an arbitrary state ρ on $A \otimes B$. Not only is this a semidefinite programme (SDP), as we shall see it shares many good properties with the von Neumann entropy.

[R. Renner, PhD thesis, arXiv:quant-ph/0512258;
M. Tomamichel, PhD thesis, arXiv:1203.2142]

[Entropy digression:

The von Neumann entropy of a state ρ is $S(\rho) = -Tr\ \rho \log \rho = H(\vec{\lambda})$, where $\vec{\lambda}$ is the eigenvalue vector of ρ.

We define information expressions by linear-combining entropies, à la Shannon:
E.g. $S(A|B)_\rho = S(\rho^{AB}) - S(\rho^{B})$, or conditional mutual information

$$I(A:B|C)_\rho = S(\rho^{AB}) + S(\rho^{BC}) - S(\rho^{ABC}) - S(\rho^{C}).$$

[See (e.g.) M. M. Wilde, QIT, Cambridge University Press]

[Entropy digression:

While the quantum conditional entropy $S(A|B)_\rho = S(\rho^{AB}) - S(\rho^B)$ can be negative, the conditional mutual information

$$I(A:B|C)_\rho = S(\rho^{AB}) + S(\rho^{BC}) - S(\rho^{ABC}) - S(\rho^C)$$

is always ≥ 0; this is Lieb-Ruskai's strong subadditivity...

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2. Min-entropy calculus

Conditional (smooth) min-entropy:

$$H_{min}(A|B)_\rho = -\log \min_{\lambda,\sigma} \mathrm{Tr}\, S \text{ s.t. } \rho^{AB} \leq 1^A \otimes S^B$$

In the applications (privacy amplification, ...) we can gain by "smoothing":

$$H^\varepsilon_{min}(A|B)_\rho = \max H_{min}(A|B)_\omega \text{ s.t. } P(\rho,\omega)\leq\varepsilon$$

Amazingly, also this is an SDP! Useful due to SDP duality (...)

[R. Renner, PhD thesis, arXiv:quant-ph/0512258;
M. Tomamichel, PhD thesis, arXiv:1203.2142]

2. Min-entropy calculus

Motivated by the von Neumann entropy identity $S(A|B) = -S(A|C)$ for a pure state on ABC, we define also a

conditional (smooth) max-entropy:

$$H_{max}^{\varepsilon}(A|B)_\rho = -H_{min}^{\varepsilon}(A|C)_\varphi \text{ w.r.t. purification } \varphi$$

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As in the classical case, "min" applies to extraction problems, while "max" to compression problems...

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M. Tomamichel, PhD thesis, arXiv:1203.2142]

2. Min-entropy calculus

These quantities share many features with the von Neumann entropy, e.g. strong subadditivity reads $S(AIBC) \leq S(AIB)$:

$$H_{min}^{\varepsilon}(AIBC)_\rho \leq H_{min}^{\varepsilon}(AIB)_\rho ;$$

$$H_{max}^{\varepsilon}(AIBC)_\rho \leq H_{max}^{\varepsilon}(AIB)_\rho .$$

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2. Min-entropy calculus

$$H_{min}^{\varepsilon}(A|BC)_\rho \leq H_{min}^{\varepsilon}(A|B)_\rho;$$

$$H_{max}^{\varepsilon}(A|BC)_\rho \leq H_{max}^{\varepsilon}(A|B)_\rho.$$

Furthermore, while H_{min}^{ε} increases and H_{max}^{ε} decreases with ε, we have for suff. small ε and δ,

$$H_{min}^{\varepsilon}(A|B)_\rho \leq H_{max}^{\delta}(A|B)_\rho + O(1).$$

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M. Tomamichel, PhD thesis, arXiv:1203.2142]

2. Min-entropy calculus

Using properties of typical subspaces, one can also show the "quantum AEP": For n-tensor power states $\rho^{\otimes n}$ on A^nB^n, and $0<\varepsilon<1$,

$$\lim_{n\to\infty} \frac{1}{n} H_{\min}^{\varepsilon}(A^n|B^n) = S(A|B)_\rho,$$

$$\lim_{n\to\infty} \frac{1}{n} H_{\max}^{\varepsilon}(A^n|B^n) = S(A|B)_\rho.$$

[R. Renner, PhD thesis, arXiv:quant-ph/0512258;
M. Tomamichel, PhD thesis, arXiv:1203.2142]

3. Applications: QKD, ...

The min-entropy formalism was initially devised to give unified and optimized security proofs of QKD protocols.

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Since then it was used in condensed matter physics [Area laws, Brandão/Horodecki] and for strong converses [talk by Ciara Morgan].

3'. LOCC hiding efficiency

Recall the state discrimination problem, but now let ρ and σ be states on a composite system: $\mathcal{H} = A \otimes B$, and let A (Alice) and B (Bob) be far apart.

Without a quantum channel between them, they can only perform *local operations and classical communication* (LOCC).

[Walgate et al., PRL85:4972-4975, 2000]
[Terhal/DiVincenzo/Leung, PRL86:5807-5810, 2001]

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$$\max P\{\text{guess}\} =: \frac{1}{2} + \frac{1}{4}\,\|\rho-\sigma\|_{LOCC}$$

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3'. LOCC hiding efficiency

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abc

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llap

guess

$$\max P\{\text{guess}\} =: \frac{1}{2} + \frac{1}{4}\,\|\rho-\sigma\|_{LOCC}$$

Can be much smaller than $\|\rho-\sigma\|_1$: data hiding!

[Terhal/DiVincenzo/Leung, PRL86:5807-5810, 2001]

3'. LOCC hiding efficiency

Let ρ_x be states on a composite system $\mathcal{H} = A \otimes B$, with the systems of Alice and Bob having dimensions $|A|=|B|=d$ ($x=1...M$), such that any pair of states is LOCC-hiding, but that there is a global POVM revealing x among all M.

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How large can M be in relation to d?

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Known for a while [e.g. Hayden/Leung/Shor/AW, CMP 250:371-391, 2004]:

Random states in dxd of rank

$r = d\,\mathrm{polylog}(d)$ are hiding states.

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Can we have more? NO!

[AW, in preparation, 2014-2015]

3'. LOCC hiding efficiency

Compare secret sharing [A. Shamir, 19791; G.R. Blakley, 1979]: In any tight secret sharing scheme (i.e. where all subsets are either authorized or adversarial), each relevant share must be at least as large as the secret.

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Here "(2,2)" - note that LOCC data hiding cannot be perfect (re secrecy).

Formal statement: If $\varepsilon, \delta > 0$ are small enough, s.t. $\|\rho_x - \rho_y\|_{LOCC} \leq \delta$ for all x, y, and $\mathrm{Tr}\rho_x D_x \geq 1-\varepsilon$ for a POVM (D_x).

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Then: μ, $C>0$ (dep. only on ε,δ) with

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Recall entropy chain rule identity

$S(ABIC) = S(AIC) + S(BIAC)$.

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Recall entropy chain rule identity $S(ABIC) = S(AIC)+S(BIAC)$. Turns into inequalities for min-/max-entropies:

$$H_{\min}^{\varepsilon+2\varepsilon'+\varepsilon''}(AB|C)_\rho \geq H_{\min}^{\varepsilon'}(A|BC)_\rho + H_{\min}^{\varepsilon''}(B|C)_\rho - \log\frac{2}{\varepsilon^2}, \quad (5.11)$$

$$H_{\max}^{\varepsilon+\varepsilon'+2\varepsilon''}(AB|C)_\rho \leq H_{\max}^{\varepsilon'}(A|BC)_\rho + H_{\max}^{\varepsilon''}(B|C)_\rho + \log\frac{2}{\varepsilon^2}, \quad (5.12)$$

$$H_{\min}^{\varepsilon+3\varepsilon'+2\varepsilon''}(A|BC)_\rho \geq H_{\min}^{\varepsilon'}(AB|C)_\rho - H_{\max}^{\varepsilon''}(B|C)_\rho - 2\log\frac{2}{\varepsilon^2}, \quad (5.13)$$

$$H_{\max}^{2\varepsilon+\varepsilon'+2\varepsilon''}(A|BC)_\rho \leq H_{\max}^{\varepsilon'}(AB|C)_\rho - H_{\min}^{\varepsilon''}(B|C)_\rho + 3\log\frac{2}{\varepsilon^2}, \quad (5.14)$$

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[A. Vitanov, MSc thesis 2011, cf. 1203.21442]

Proof (sketch):

Define $\Omega^{XAB} = \frac{1}{M} \sum_x |x\rangle\langle x|^X \otimes \rho_x^{AB}$. Then:

$$\log(M-1) \leq H_{min}^{\sqrt{\delta}}(X|B)$$

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$$\leq H_{\max}^{\mu}(A|B) + O(1)$$ {Decodability}

QED

Compare secret sharing [A. Shamir, 19791; G.R. Blakley, 1979]: In any tight secret sharing scheme (i.e. where all subsets are either authorized or adversarial), each relevant share must be at least as large as the secret.

Corollary: In any tight LOCC data hiding scheme, each relevant share must be at least as large as the secret (plus a constant). Attained for "(n,t)" scheme.

[AW, in prep.; Hayden/Leung/Smith, PRA 71:062339, 2005]

Thank you for your attention