Satisfying demands in a multicellular network: A universal power allocation algorithm

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1. Introduction

Multi-input multi-output (MIMO) combined with network densification promise improved network coverage and capacity for mobile broadband access. But, due to an increased number of transmit antennas and or the proximity of base stations (BS), users at cell edges experience a higher degree of interference from neighboring base stations.

Network MIMO or other forms of BS co-operation enable sharing complete or statistical knowledge of channel states (CS) amongst neighbors via back-haul links to alleviate interference and offer better rates to users. When back-haul is not available, each BS may estimate the local channel state information and use the same for better performance. In some cases, a low rate feedback from the receiver indicating the QoS of the current transmissions is utilized, while in the worst case the transceivers are designed with no CS information. Thus we have a variety of systems with varying degrees of the information about the interfering channels. However the goal in each is the same: satisfy the demands of all the users. We may require higher power profiles to satisfy the same demands when working with lesser information. Further diverse situations can arise because of the system configuration like modulation, precoding, channel coding, resource allocation etc.
For a given vector of power constraints at various base stations, Shannon capacity gives the maximum achievable rate, i.e., the capacity region. This is an upper bound. We define “system specific capacity region” (achievable rate region of a given system) which depend on coding (space–time, channel), modulation, channel state information availability, synchronization, feedback errors and many other things. Given a system architecture with a chosen set of parameters which define its rate allocation, modulation, etc., the achievable rates are usually inferior to the theoretical rates and the system specific capacity region is defined based on these rates. The system-specific capacity region for the same power constraint varies: for example it shrinks if the number of supported discrete rates reduce. Thus, the power allocated to any user to achieve the same demand rate varies with the set of system parameters.

The main contribution of this paper is a universal algorithm which can work with many of the systems mentioned above. It satisfies asymptotically the demands of all the users irrespective of the system in which it is operating, albeit with different power profiles. Each base station requires minimal information: its user’s demands, its total power constraint and the current transmission rates to its users. The amount of data information transmitted successfully in a slot (per slot) basically represents the current transmission rates. These current transmission rates are decided by the serving base stations either using complete CSIT (algorithm can also be used as a centralized scheme in this case) or has to be estimated completely blindly or using some partial information. These are also influenced by the underlying channel.

In cellular networks, the scenarios can change with time. For example a base station can become active suddenly, the demands may change etc. We demonstrate via simulations that the proposed algorithms can also track the changes.

In heterogeneous networks, various agents (for example macro cells and micro cells) can operate at different speeds. However they still can interfere with each other. We consider a simple example scenario and demonstrate using the two time scale stochastic approximation analysis that the proposed algorithm converges to the same power profile irrespective of the disparities in the update rates. We then illustrate the same for general scenarios using numerical simulations. The following are the contributions of this paper:

1. A system specific game theoretic problem formulation using the system specific capacity region.
3. Various properties (e.g. convergence) of the proposed algorithm is analyzed using an ODE framework.
4. Simulation results demonstrate the effectiveness of the proposed algorithm for a variety of systems.
5. We also establish the tracking capabilities of the algorithm.
6. We illustrate the robustness of the proposed algorithm against the disparities in update rates at various agents.

1.1. Related work

For an excellent survey on power control in wireless networks, the reader is referred to [2] and the references there-in (e.g. [3,5–8]). In recent years, several authors have addressed distributed power control strategies with various levels of co-operation for a given system configuration (e.g. [3,5–7,10] etc). Typically, the design objective is to maximize the total sum rate of all the users subject to BS power constraints or to minimize the total transmit power satisfying some SINR constraints of the users.

Most of the existing algorithms aim at either optimizing the total power spent keeping the QoS above a required level (e.g. [5–7] etc.) and or optimize the QoS while keeping the power utilized within a given budget (e.g. [10]). But our algorithm does not optimize, it only meets the demands (in the form of average transmission rates) on average asymptotically.1 This relaxation helps us in proposing an algorithm that requires minimal information (hence has minimal complexity) at the transmitters: rates at which the information is correctly transmitted to the user in every slot. Data is pumped out from the transmitter and hence these rates are readily known to the transmitter. Hence this algorithm does not require any extra information and this can be exploited in many more ways. For example, one can probably use this algorithm in networks with heterogeneous cells, i.e., when each cell has a system configuration that can be different from the other cells.

A related concept, called satisfying equilibrium, is defined and studied in a recent paper ([9]). Here they define the satisfying equilibrium as any profile at which the QoS of all the users is either better or the same as the specified level. Basically, the set of satisfying equilibrium represents the domain of optimization for the problems that optimize the total power utilized while maintaining the QoS. In our paper, we propose an algorithm that satisfies the demands for all the users at exactly the specified level via a stochastic approximation based zero finding method. As already discussed, this zero finding method greatly simplifies the algorithm. To the best of our knowledge this is the first paper that proposes to take advantage of the relaxation obtained by avoiding the optimization.

1.2. Organization

We introduce the system model in Section 2. In Section 3, we describe the system specific problem formulation. The algorithm and its analysis is presented in Section 4. Section 5 provides simulations. Section 6 discusses heterogeneous agents. Appendix contains example systems and proofs.

1.3. Notations

Boldface lower-case symbols represent vectors, capital boldface symbols denote matrices (\(I_n\) \(\in\) \(\mathbb{N} \times N\) identity matrix). Hermitean transpose is denoted \(^H\) while \(\text{tr} [\mathbf{X}]\) represents the trace of matrix \(\mathbf{X}\). All logarithms are base-2 logarithms. Small letters represent the scalars. Let \(a_k\) represent the kth component of the vector \(a\). If the vector is already indexed like for example in \(p_k\) then \(p_k\) represents its kth component. Let \((p_s)\) represent the component-wise product, i.e., \((p_s)_k = p_k s_k\) for all \(k\) while \(\sqrt{p}\) represents component wise square root. \(E[\cdot]\) denotes expectation and \(E_s\) is expectation w.r.t to \(s\) when conditioned (if any) on the other random variables.

2. System model

We consider a multi-cell MIMO system. Each base station has \(M\) transmit antennas and is communicating with \(K\) single-antenna users (see Fig. 1). Every user experiences both intra-cell (transmissions from parent BS) and inter-cell (transmissions from neighboring BS) interference. Each user in a cell demands a certain rate and all these rates have to be jointly satisfied by the BS (present in the cell) while operating within a total power constraint.

Let \(H_{ij}\) represent the \(K \times M\) channel matrix, when the users in cell \(j\) receive signals from the BS of cell \(i\) and let its elements be given by zero-mean unit-variance i.i.d. complex Gaussian entries. Let \(n_j\) represent the additive white Gaussian noise at the receivers of

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1 We show the demand meeting power profile to be a NE of a ‘leaky’ game. We call this game ‘leaky’, because the utility of the game is upper bounded by the demands (see Definition 5, Section 3.1). In summary our aim is to provide a channel, to each one of the users, whose (system specific) capacity is more than or equal to the user’s demand.
cell \(j\). \(\mathbf{x}_j\) be the \(M\) length transmit vector in cell \(j\) and \(\gamma_j \in [0,1]\) be
the interference factor, representative of the level of interference from cell \(l\). For example, as base stations become denser, interfer-
ence increases and hence \(\gamma_j \rightarrow 1\). The signal vector (of length \(K\)) received by users in cell \(j\) is given by,

\[
y_j = \mathbf{H}_j \mathbf{x}_j + \sum_{l=1 \neq j}^N \gamma_j \mathbf{H}_l \mathbf{x}_l + \mathbf{n}_j \quad \text{for all} \quad j \leq N. \tag{1}
\]

In the above the first term represents the useful signal part as well as the intra-cell interference while the second term (summation) represents the inter-cell interference to the \(j\)th cell from its neighbors.

If \(\mathcal{P}_j\) represents the total power constraint in cell \(j\), then \(\text{tr}(\mathbb{E}[\mathbf{x} \mathbf{x}^H]) \leq \mathcal{P}_j\) to satisfy the power constraint. As an example, if the BS in cell \(j\) uses power levels specified by \(\mathbf{p}_j\) and a precoding matrix \(\mathbf{G}_j\) (of size \(M \times K\)), then the transmit vector is given by \(\mathbf{x}_j = \mathbf{G}_j(\sqrt{\mathbf{p}_j} \mathbf{s}_j)\) (\(\mathbf{s}_j\) is a \(K\) length independent symbol vector of zero mean and unit variance components). In this case the power constraint leads to,

\[
\text{tr}(\mathbb{E}[\mathbf{x} \mathbf{x}^H]) \leq \text{tr}(\mathbb{E}[\mathbf{G}_j \sqrt{\mathbf{p}_j} (\mathbf{G}_j \sqrt{\mathbf{p}_j})^H]) \leq \mathcal{P}_j \quad \text{for any} \quad j.
\]

Given a precoding scheme, this constraint can equivalently be represented by (for a possibly different \(\mathcal{P}_j\)) \(\sum \gamma_j \mathbf{H}_l \mathbf{x}_l \leq \mathcal{P}_j\). The symbol, \(u_{kj}\), received by the user \(k\) of cell \(j\) is

\[
y_{kj} = h_{kj}^H \mathbf{x}_j + \sum_{l=1 \neq j}^K h_{kl}^H \mathbf{x}_l + \sum_{i=1 \neq j}^K \gamma_i h_{ij}^H \mathbf{x}_i + n_{kj} \\
= u_{kj} + h_{kj} + \sum_{l=1 \neq j}^K h_{kl} + n_{kj} \tag{2}
\]

where \(h_{kj}\), the \(k\)th row of matrix \(\mathbf{H}_j\), represents the \(M\) length channel vector for user \(k\) of cell \(j\) as received from the BS of cell \(l\). In the above, \(u_{kj}, h_{kj}\) and \(h_{kl}\) respectively represent the useful, intra-cell interference and inter-cell interference signal.

2.1. System with no precoding

This paper proposes an algorithm which works for any system in general. By system, we mean a particular multi-cell network with a given configuration like, precoding scheme, channel coding, resource allocation etc. We will derive the exact received signal characteristics for one such example system. The received signal characteristics of the others system can be derived in a similar way. We consider a system with no precoding (for example, systems which does not have access to channel state information). Further we consider a system with \(M = K\) and with \(\mathbf{x}_j = (\sqrt{\mathbf{p}_j} \mathbf{s}_j)\). The average power in the useful, intra-cell, inter-cell interference signals of the received signal (after channel coding at the transmitter and channel decoding at the receiver) after averaging w.r.t. to the symbol statistics \(\{s_j\}\) for any given channel state is:

\[
\begin{align*}
\mathbb{E}_{\{u_{kj}\} \mid \{h_{kj}\}}[u_{kj}^2] &= p_{kj}\|h_{kj}\|^2; \\
\mathbb{E}_{\{u_{kj}\} \mid \{h_{kj}\}}[h_{kl}^2] &= \sum_{j \neq k} p_{kj}\|h_{kl}\|^2 \quad \text{and} \tag{3} \\
\mathbb{E}_{\{u_{kj}\} \mid \{h_{kj}\}}[n_{kj}^2] &= \sum_{k \neq j} \gamma_k p_{kj}\|h_{kj}\|^2
\end{align*}
\]

where, \(h_{kj}\) is the \((k,j)\)th component of the matrix \(\mathbf{H}_j\). In the above we used \(\mathbb{E}[s_k, s_j'] = \mathbb{1}_{|k-j'|}\).

3. System specific problem formulation

Every BS has to meet its users demands, for example BS \(j\) has to meet its users demand rates represented by \(r_j := \{r_{kj}, k \leq K\}\). It has to tune its power levels \(\mathbf{p}_j\) to achieve this. But the rates achieved will also depend upon the powers used by the other base stations. Our goal is to find a simple universal power allocation algorithm which runs independently and simultaneously at all the base stations and tunes the power levels to achieve the user demands using minimal information. The power levels depend upon the system configuration (for example channel precoding scheme, rate allocation scheme). We consider some interesting example systems briefly in Table 1 and described in Appendix A. These systems are referred using a three part code, I–II–III, as explained below (see Table 1):

(1) The first part (I) represents the availability of channel state information at transmitter\(^2\): (a) A represents an asymptotic (large number of antennas/users) system, where achievable rates for almost all CS are approximated by a constant (see [11] and references there in), (b) C for systems with complete CSIT, (c) L, systems with local CSIT, i.e., BS \(j\) knows \(\mathbf{H}_j\) part of the CS, (d) N, systems with no CSIT.

(2) The second part (II) represents the transmission rates used at the system\(^3\): (a) I for ideal systems which can channel code to achieve any feasible rate, (b) D for those systems which can only operate at one of the discrete rates in the set \(\mathbb{R} = \{r_1, r_2, \ldots, r_{N_b}\}\) (arranged in decreasing order), (c) RA for

<table>
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<tr>
<th>CSIT</th>
<th>TX rate</th>
<th>Precoder</th>
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<tr>
<td>A</td>
<td>Asymptotic</td>
<td>I</td>
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<td>C</td>
<td>Full CSIT</td>
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<tr>
<td>L</td>
<td>Local CSIT</td>
<td>RA</td>
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<tr>
<td>N</td>
<td>No CSIT</td>
<td>RAE</td>
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Table 1: System specification [I–II–III].

\(^2\) One can also consider systems which have an estimate of the CS.

\(^3\) We illustrate these concepts using simple rate allocation schemes. One can extend it to other rate allocations, for e.g. schemes that incorporate fairness.
systems which estimate the current rate (i.e., pick the current maximum possible rate from the set $\mathbb{R}$ as explained in Appendix A) using rate adaptation schemes (e.g. [12]) without CSIT, (d) RAE when there are estimation errors in the rate adaptation algorithm (see systems N-RA-NO and L-RAE-ZF of Appendix A for more details).}

(3) The third part (III) represents the *precoder*: (a) ZF for zero forcing precoding, (b) NO for no channel precoding.

### 3.1. Game theoretic formulation

As the base stations influence each other, the problem can best be captured using a game theoretic formulation. We begin by introducing the components of the game. The calligraphic letters (for example $\mathcal{P}$) represent the ensemble of either vectors, matrices or scalars for all the base stations.

**Power profile**, $\mathcal{P} := \{p_{kj}\}_{k \in \mathcal{K}, j \in \mathcal{N}_k}$, represents the vector comprising of the powers used at all the base stations for all the users. Recall, $p_{kj}$ represents the power used by the BS of cell $j$ for user $k$ in cell $j$.

**Channel State (CS)**, $\mathcal{H} := \{H_1, H_2, \cdots, H_N\}$, arranged as a matrix of dimension $KN \times MN$, represents the channel state of the entire system.

**Rate for a given power profile and system**, $R_{\text{sys}}^{\mathcal{P}}(\mathcal{P}, \mathcal{H})$, represents the transmission rates allocated, to the user by the base station $j$, in system represented by $\mathcal{S}$ (e.g. N-RAE-NO in Table 2) when the base station uses powers $\mathcal{P}$ and when the CS is $\mathcal{H}$. These rates are given in the right column of the Table 2 for various example systems, whose detailed descriptions are provided in Appendix A.

**Average Rate for a given system and power profile**, is the rate that is achieved on average when a given system uses the power profile $\mathcal{P}$: $R_{\text{sys}}^{\mathcal{P}}(\mathcal{P}) = \mathbb{E}[R_{\text{sys}}^{\mathcal{P}}(\mathcal{P}, \mathcal{H})]$. Let $R_{\text{sys}}^{\delta_{\mathcal{P}}} := \{R_{\text{sys}}^{\delta_{\mathcal{P}}} \Delta_{\mathcal{P}} \}_{\Delta_{\mathcal{P}}}$. **Power constraint** ($\mathcal{P} \leq \overline{\mathcal{P}}$) We use $\leq$ in a special manner to facilitate defining the power constraint. We say a power profile $\mathcal{P}$ is “less than or equal to” and hence satisfies the constraint defined in terms of another power profile $\mathcal{P}'$ if the two profiles satisfy the constraints for each base station as: $\sum_k p_{kj} \leq \sum_k p_{kj}'$ for all $j \in N$.

**System Specific Capacity Region** for any given power profile constraint $\mathcal{P}$ and a system, $\mathcal{S}$, is defined as the collection of all possible tuple of average rates achievable, while using powers that satisfy the constraints defined in terms of $\mathcal{P}$, i.e.,

$$\mathcal{C}_y^{\mathcal{P}}(\mathcal{P}) := \{ (R_{kj})_{j \in \mathcal{N}_k} \in \mathbb{R}^N : \text{for all } k \mathcal{R}_{kj} = R_{\text{sys}}^{\delta_{\mathcal{P}}}(\mathcal{P}) \}$$

for some $\mathcal{P}$ with $\mathcal{P} \leq \overline{\mathcal{P}}$. (4) This region is different for different systems. For a system with ideal rates the capacity region coincides with the theoretical one. A system with discrete rates cannot always achieve the maximum possible rate and hence its capacity region shrinks. It further depends upon $\mathbb{R}$, the set of supported rates. If the system has estimation errors, the capacity region shrinks further.

**Utilities and Players**: Each BS $j$ is a player and its strategy is $K$-dimensional power vector, $\mathcal{P}_j := \{p_{1j}, \ldots, p_{Kj}\}$. Note that $\mathcal{P} = [p_1, p_2, \ldots]$. Define the utility of player $j$ as,

$$U_j(y)(\mathcal{P}_j, \mathcal{P} - \mathcal{P}_j) = \min_{\mathcal{P} - \mathcal{P}_j} \left\{ \mathbb{E}\left[ R_{\text{sys}}^{\delta_{\mathcal{P}}}(\mathcal{P}_j, \mathcal{P}_j, \mathcal{P}_j, \mathcal{P} - \mathcal{P}_j) \right] \right\}$$

with $\mathcal{P}_j := \{p_1, p_2, \ldots, p_{j-1}, p_{j+1}, \ldots, p_N\}$. (5)

In the above, $\mathcal{P}_j$ is the power vector profile excluding only the powers of BS of cell $j$ and $\mathcal{R}_{kj}$ is the demand of user $k$ of cell $j$. Every system with given power constraint $\mathcal{P}$ and demand vectors $(\mathcal{R}_{kj})$ defines an $N$-player non cooperative strategic form game: $((K, 2 \cdot \cdots \cdot 2), \{U_j\}_{j \in \mathcal{N}_k})$. The Nash equilibrium (NE) of this game is a power profile $\mathcal{P}$ that satisfies,

$$\mathcal{P}_j \in \arg\max_{\mathcal{P}_j \in \mathcal{P}} U_j(y)(\mathcal{P}_j, \mathcal{P} - \mathcal{P}_j)$$

for all $j$. (6)

From the above definitions, it is evident that,

**Lemma 1**. For any given system and power constraints $\mathcal{P}$, if the vector of the demands $(\mathcal{R}_{kj})_{j \in \mathcal{N}_k}$ is in the corresponding capacity region $\mathcal{C}_y^{\mathcal{P}}(\mathcal{P})$, then there exists a $\mathcal{P} \leq \overline{\mathcal{P}}$, which is a NE satisfying all the demands:

$$R_{\text{sys}}^{\delta_{\mathcal{P}}}(\mathcal{P}) = \mathcal{R}_{kj}$$

Thus, when all the base stations use the NE power profile $\mathcal{P}$ of Lemma 1, all the users in each cell achieve an average rate which equals their demand, i.e., will be able to receive the information

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4 One can also consider other types of precoders (e.g. MMSE). Our analysis and proofs hold for these configurations as long as they satisfy the assumptions A.1 to 4 (refer Section 4).

5 The utility of an user is the average rate at which its data is transferred. The user $k$ of cell $j$ requires transmission at maximum at its demand rate $r_{kj}$ and hence his utility is upper bounded by the same.
A.3 The instantaneous rates are bounded by the same constant, i.e., $|R_{jk}^\theta(P, \mathcal{H})| \leq B$ for all $k, j, P$ and $\mathcal{H}$.

A.4 The average rate $R_{\text{avg},jk}^\theta$ is continuous in $P$ for all $k, j$.

We will show that the UPAMCN trajectory (7) can be approximated by the solution $(P(t))$ of the following ODE (to be precise a differential inclusion).

$$\dot{p}_{jk} = r_{jk} - R_{\text{avg},jk}(P) + z_{jk}(P) \quad \text{for all } k, j$$

(8)

where $z_{jk}(P)$ represents the projection term. Define the limit set of this ODE:

$$L^\text{ODE} := \lim_{t \to \infty} \mathcal{P}(s) : s \geq t \text{ and } P(0) = P.$$

The $\delta$-neighborhood of this set is defined as:

$$B_\delta(L^\text{ODE}) := \{ P : |P - \mathcal{P}| \leq \delta \text{ for some } \mathcal{P} \in L^\text{ODE} \}.$$

Theorem 1 establishes that the trajectory ultimately spends time in this limit set. We first establish the theorem and later study the systems of previous section using this limit set.

Theorem 1. Assume A.1–4. Then for every $\delta > 0$, the fraction of time the tail of the algorithm (for any initial power profile with $\bar{P} < P$) spends in the $\delta$-neighborhood of the limit set $B_\delta(L^\text{ODE})$ tends to one (in probability) as $t \to \infty$.

Proof. Refer Appendix B. $\square$

4.3. Analyses of the specific systems

Most of the systems considered in this paper (for example, C-D-ZF) transmit at one of the rates from a discrete set $\mathbb{R}$ depending on the instantaneous CS and for these one need to explicitly prove the continuity of the average rates. This is achieved in the following (proof in Appendix B).

Lemma 2. Assume A.1 and A.2. Then, for all the systems considered in Table 2, assumptions A.3 and A.4 are satisfied, Theorem 1 applies and hence the UPAMCN trajectory (7) asymptotically spends most of its time in the limit set, $L^\text{ODE}$. Further, the demand meeting NE set, $L^\text{NE}$, is non empty and these form the stationary points of the ODE (8), whenever for all $k, j$ the demands satisfy

$$r_{jk} \leq \sup \ R_{\text{avg},jk}(P).$$

For further analysis, one needs to study the limit set of ODE (8). A limit set of an ODE usually contains limit cycles or attractors. Every demand meeting NE of $L^\text{NE}$ is a stationary point of the ODE, whenever it is not on the boundary. The demand meeting NE of $L^\text{NE}$ would be in the limit set, if further, we could show that they are attractors. In that case, the algorithm spends most of its time in these attractors or in other words the UPAMCN algorithm asymptotically meets the demands of all the users. Right now, we can only say that, every stationary point of ODE (8) is a demand meeting NE and any attractor of the ODE must be a stationary point. We will show via numerical examples in the next section that the algorithm indeed converges to a demand meeting NE for all the systems considered in this paper.

4.4. Extensions to UPAMCN

The UPAMCN algorithm works under the basic assumption that the BS always has sufficient data to transmit. But in reality, data at the demand rate on average. The main aim of this paper is to obtain this NE (time) asymptotically (if required in a completely distributed way) for any given system when the demands satisfy Lemma 1. This NE depends on the system considered (for example higher amount of power may be required if one uses discrete rates in the place of ideal rates) even if the power constraint and demands are same. The proposed algorithm is a general iterative algorithm which works irrespective of the system considered, i.e., the proposed algorithm converges to the system specific NE.

Remark on hypothesis of Lemma 1: It requires that the demands equal one of the average rates of the capacity region. Lemma 2 of the next section gives an easily verifiable assumption which ensures this hypothesis of Lemma 1.

Set of demand meeting NE, $L^\text{NE} \subseteq \mathbb{R}^P$, is the set of NE which meet the demands as in Lemma 1.

We now present the Universal Power Allocation algorithm for power constrained Multi Cell Networks (UPAMCN).

4. Universal algorithm: UPAMCN

We consider a quasi-static channel and obtain the NE of Lemma 1 asymptotically by iterating the power profile at the beginning of every slot, during which the CS is assumed constant.

Basic idea\footnote{Most of the stochastic approximation algorithms obtain optimum of a function as the zero of its derivative. In contrast, this algorithm obtains the profile that satisfies the demands, directly as the zero of the function given by the average rate minus demand. In other words, this algorithm does not perform any optimization, but rather just finds a power profile that solves the equation average rate = demand for all the users.}: Each BS $j$ in every time slot knows the rates at which data is transmitted to its users, $\left\{ R_{jk}^\theta(P, \mathcal{H}) \right\}_{P, \mathcal{H}}$. The characterization of these rates is provided for some examples in Table 2 (and details in Appendix A). An iterative algorithm can find the average value of it. One can then update the power vectors to force this average towards the demands $\{ r_{jk} \}$.

Let $d_{jk}^t$ represent the number of bytes of data transmitted successfully in time slot $t + 1$ by the $j$th base station to its user $k$ divided by the duration of the time slot. This ratio depends upon the power profile of the entire system in the previous slot ($P^t$) and the entire CS in the current slot ($\mathcal{H}^{t+1}$), but ($P^t$, $\mathcal{H}^{t+1}$) are only partially known at the base stations. However $d_{jk}^{t+1}$ is still known at base station $j$ as it is the source that pumps out the data. In fact, it will be precisely equal to $d_{jk}^{t+1} = R_{jk}^\theta(P^t, \mathcal{H}^{t+1})$ of Table 2 by definition. Let $\{ \mu t \}$ represent the update step sizes.

4.1. UPAMCN algorithm

With $P_t$ representing the projection into the set $\mathcal{A}$, the UPAMCN algorithm is given by

$$P_{t+1}^{\text{UPAMCN}} = \Pi_{\mathcal{A}} \left\{ p_{jk}^t - \mu t \left( d_{jk}^t - r_{jk} \right) \right\} \quad \text{with} \quad \mathcal{A} := \left\{ p \in \mathbb{R}^K : \sum_k p_k \leq B \right\},$$

$$A_j := A_1 \times A_2 \cdots \times A_N.$$  

(7)

4.2. Analysis

We obtain the asymptotic analysis of the algorithm using the ordinary differential equations (ODE) approach of [1]. We establish Theorem 1 given below, under the following conditions:

A.1 There exists a sequence $\mathcal{Z}_t \to \infty$ with $\lim_{t \to \infty} \sup_{0 \leq t < \infty} \mu_t = 0$.

A.2 The channel state $\{ \mathcal{H}_t \}$ is an independent and identically distributed (IID) sequence with finite mean and variance.
often arrives in real time and hence there can be situations when the BS can transmit at a higher rate but does not have sufficient data. In this case we propose the following extension to UPAMCN:

\[
b_{k_j}^{t+1} = b_{k_j}^t + B_{k_j}^{t+1} - \min \left\{ d_{k_j}, b_{k_j}^t \right\} \quad \text{and} \\
\rho_{k_j}^{t+1} = \Pi_{\alpha_j} \left[ \rho_{k_j}^t + \mu_t \left( \min \left\{ d_{k_j}, b_{k_j}^t \right\} - r_{k_j} \right) \right]
\]

(9)

where \( b_{k_j}^t \) represents the remaining (accumulating) bytes of data to be transmitted by BS \( j \) to the user \( k \) at the beginning of time slot \( t \) and \( B_{k_j}^{t+1} \) represents the fresh sample of data added to the corresponding buffer.

5. Numerical simulations

We consider two types of cellular networks in our simulations (Table 3). The first one is a Hexagonal network, where users in each cell experience interference from BS transmissions of surrounding cells (typically assumed to be from the 1st tier of surrounding 6 cells) [see for example Fig. 1]. The second one is a linear network, where users in each cell experience interference from BS transmissions of adjacent cells (typically two adjacent neighbors). The system configurations are summarized in Table 4. Each BS equipped with \( M \) transmit antennas is serving \( K \) users in its cell. In all the simulations we also compute the average rates via the following iteration:

\[
\phi_{k_j}^{t+1} = \phi_{k_j}^t + \mu_t \left( d_{k_j}^t - \phi_{k_j}^t \right)
\]

for all \( k, j \). This iteration is only a measurement procedure that is used for the purpose of calculating average rates of the system for the numerical examples considered. How it represents the average rate can be understood by noticing that, \( \phi_{k_j} \) is actually a weighted average of all the instantaneous rates \( d_{k_j}^t, t \leq t \) up to time \( t \). These average rates are used to illustrate that systems considered in these examples, asymptotically (as time progresses) satisfy the user’s demands on average.

The power limit on each BS is set to 1 unit. Interference factor \( \gamma \) from each interfering BS, \( I \), is set to 0.5. For the simulations considered here, we choose the demand rate vector (to lie within the capacity region and is common for all the base stations) as:

\[
r = [0.065, 0.130, 0.195, 0.260, 0.325, 0.389, 0.454, 0.520].
\]

In the first set of simulations, we consider the hexagonal network (H1). The rate and the power convergence behavior of the algorithm for systems S1, S2 and S3 is plotted in Figs. 2 and 3, respectively. We observe that: (1) The algorithm converges to the demand meeting NE: we see in Fig. 2 that for all the systems, the average rate achieved asymptotically converges towards the demand rates (the dotted lines). (2) As discussed in the previous sections, we notice from Fig. 3, that the converged power profile (demand meeting NE) is system specific. S3 is a system with errors, the proposed algorithm still satisfies the demands asymptotically, however, the converged power profile has higher power levels in comparison with the error free systems S2 and S1. (3) Note that S2 can also represent C-D-ZF, a complete CSIT system (see details on Table 2 and Appendix A). From Fig. 3, we observe that the converged power profile of C-D-ZF (S2) is close to that of A-I-ZF (S1) system. Thus the demand meeting power profile of systems with large number of transmit antennas and or users and large number of discrete levels in \( R \) is close to that of the asymptotic ideal rate system. Further convergence is faster with S1 system. Thus for such systems, UPAMCN algorithm can be used to estimate (approximately) the demand meeting power profile, much faster, using the asymptotic rate expressions in place of instantaneous transmit rates allocated, \( \{d_{k_j}^t\} \). Note that this further avoids the need of complete CSIT, as we need only local CSIT for precoding. (4) As the discrete levels increase, the power profile decreases and finally converges to that of the ideal rate. This is tabulated in Table 5.

In the second set of simulations, for given demand rates, we compare the algorithm behavior for different network configurations L1, L2 and H1 with system S2 (see Fig. 4). We observe that: to satisfy the same demands, the base stations in L2 expend the least power, followed by L1 and then H1. L2 performs better than L1 due to improved transmit diversity. H1 is the worst (larger number of interfering base stations).

In the final set of simulations, for network configuration H2, we consider the least informed (No CSIT) systems, the rate adaptation systems S4 and S5. We choose the common demand rate vector as \( [0.1694, 0.1936] \). Figs. 5 and 6 illustrate the average rate and power profile convergence. As CSIT (even local) is not available at the base

![Fig. 2. Rate convergence for Systems S1, S2 and S3 (H1 network).](image_url)

![Fig. 3. Power convergence for Systems S1, S2 and S3 (H1 Network).](image_url)

<table>
<thead>
<tr>
<th>Table 3</th>
<th>Network configurations.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Type</td>
<td>Inf BS</td>
</tr>
<tr>
<td>L1</td>
<td>Linear</td>
</tr>
<tr>
<td>L2</td>
<td>Linear</td>
</tr>
<tr>
<td>H1</td>
<td>Hexagon</td>
</tr>
<tr>
<td>H2</td>
<td>Hexagon</td>
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<table>
<thead>
<tr>
<th>Table 4</th>
<th>System configurations</th>
</tr>
</thead>
<tbody>
<tr>
<td>S1</td>
<td>Asymptotic Ideal with ZF Precoder (A-I-ZF)</td>
</tr>
<tr>
<td>S2</td>
<td>Rate adaptation with local CSIT and ZF precoder (L-RAE-ZF)</td>
</tr>
<tr>
<td>S3</td>
<td>Rate adaptation with local CSIT, ZF Precoder and with estimation errors (L-RAE-ZF)</td>
</tr>
<tr>
<td>S4</td>
<td>Rate adaptation without CSIT (N-RA-NO)</td>
</tr>
<tr>
<td>S5</td>
<td>Rate adaptation without CSIT and estimation errors (N-RAE-NO)</td>
</tr>
</tbody>
</table>

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Table 5
System S2.

<table>
<thead>
<tr>
<th>System</th>
<th>Demand satisfying NE (converged power)</th>
</tr>
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<tbody>
<tr>
<td>ID</td>
<td>0.015 0.032 0.049 0.067 0.085 0.105 0.126 0.147</td>
</tr>
<tr>
<td>D1</td>
<td>0.017 0.035 0.053 0.072 0.092 0.113 0.135 0.158</td>
</tr>
<tr>
<td>D2</td>
<td>0.031 0.049 0.071 0.097 0.121 0.146 0.174 0.203</td>
</tr>
</tbody>
</table>

ID: Ideal, D1: Discrete-100 levels, D2: Discrete-20 levels.

In cellular networks, often the scenario changes. A user might complete his call and depart, base station can become inactive (ex., in small cell networks some of the base stations might be switched off to conserve energy), the demands of a user can change, user position or user-BS associations can vary and so on. Like most of the stochastic approximation algorithms, even UPA-MCN can track the changes in such scenarios, if the update coefficient $\mu_t$ converges (as $t \to \infty$) to a strictly positive value. In this subsection, we provide some simulation examples to show that the proposed algorithm effectively tracks the dynamics in the system.

In Figs. 7 and 8, we consider one such dynamic scenario. Assume that a single cell (BS1) is active to begin with. After a while, a neighboring cell (BS2) becomes active and a while later another cell (BS3) becomes active. We consider 8 users in each cell, with the respective demands given by $r_1 = [0.132, 0.364, 0.528, 0.660, 0.792, 0.924, 1.056]$ (for users in first cell), $r_2 = [0.099, 0.198, 0.297, 0.396, 0.495, 0.594, 0.693, 0.792]$ (users in second cell) and $r_3 = [0.066, 0.132, 0.198, 0.264, 0.330, 0.396, 0.462, 0.528]$. The UPA-MCN algorithm initially converges to satisfy the demands of the users of the first cell, then it starts running even at the second cell and the power levels in both the cells adjust to meet their respective users demands and finally when the third cell becomes active the power levels adjust again. We notice that at the time instances when a new BS becomes active, the demands are not met, the rates diminish, but, shoots back within a short time interval (observe the notches in Fig. 8). In all, we observe that the algorithm tracks the dynamics in the multicell system and satisfies the user demands as long as the rates lie within the capacity region (see Fig. 8), albeit with different power levels in different situations (see Fig. 7).

In another scenario Figs. 9 and 10, before BS2 becomes active, the demand rates of users 4 and 7 in BS1 change (observe the demand rates changing around 3000 iterations in Fig. 10). The new rate vector is given by $r_1 = [0.132, 0.364, 0.554, 0.660, 0.792, 0.878, 1.056]$. Again we observe that the algorithm tracks these changes by suitably adjusting the power levels (Fig. 9) to meet the new demands (Fig. 10).

6. Heterogeneous agents

Often a multicell system consists of a heterogeneous network, which might support different agents with different purposes, who nevertheless interfere with each other. For example, sometimes the Macrocell sites are overlaid with Small cells (Pico and Femto cells) to fill the coverage gaps and enhance the capacity of the network (see Fig. 11). Typically these small cells, comprising of portable base stations are deployed in residential homes, super markets, office spaces and hot spots to name a few. This is just one example of a multi-tier network and one might find many such scenarios. These heterogenous interfering agents still have to
satisfy their respective users demands. A power allocation algorithm, like the one proposed in the previous sections can probably be used again to satisfy the demands. However, these heterogeneous agents need not be synchronized, may not have the same processing capabilities and hence their power profiles need not be updated at the same rate.

In this section, we study the effects of some of these disparities. We will show that the UPAMCN algorithm converges to the same

---

**Fig. 6.** Power convergence for Systems S4 and S5 (H2 Network).

**Fig. 7.** Power convergence as new cells are activated.

**Fig. 8.** Rate convergence as new cells are activated.
We demonstrate this analytically via a simple example using two time scale (stochastic approximation) analysis, while the same is shown for more practical scenarios via numerical examples.

via UPAMCN algorithm (Eq. (7)), all the agents (for example base station j) update their own components (e.g. \( p_t^j = \{ p_t^{i,j} \}_k \)) in a decentralized manner, i.e., without the requirement of the other agents updates (e.g. without \( p_t^{j} = \{ p_t^{i,j} : j \neq j \} \)) and this is true for all time \( t \). Basically the agents (e.g. agent j) observe the estimates required for their own iteration (e.g. \( \{ d_t^{i,j} \} \)) directly, avoiding the need for the knowledge of the other's parameter (e.g. \( p_t^{j} \)) thus leading to a distributed algorithm. However, it still needs the various agents to be synchronized, i.e., it needs that for every base station \( j \) the update co-efficient \( \mu \) is same and depends only upon time, i.e., at time \( t \) it equals \( \mu^t \) for all \( j \) (see Eq. (7)). In a heterogeneous network, various agents may update their components at different rates. For example, the macro cells can operate at a much slower rate than the small cells or vice versa. That is, (for example) there can be some index \( N_1 \) such the agents with \( j < N_1 \) update at every time step while the agents with \( j > N_1 \) update only once in \( \kappa \) time steps:

\[
\begin{align*}
\Pi_{n} p_k^{j} & = \Pi_{n} \left[ p_k^{j} - \frac{\mu}{\kappa} \left( d_t^{j} - r_t \right) \right] \quad \text{when} \ j < N_1 \\
p_k^{j+1} & = \Pi_{n} \left[ p_k^{j} - \frac{\mu}{\kappa} \left( d_t^{j} - r_t \right) \right] \quad \text{if} \ j > N_1, \ t = \kappa n \text{ for some } n \in \{1,2,\ldots\} \\
& \quad \text{else}.
\end{align*}
\]

6.1. Analysis using Two time scale algorithms

Stochastic approximation algorithms can be described in a general setting by:

Fig. 9. Power convergence as new cells are activated and as demands (of User 4 and 7 of BS1) change.

Fig. 10. Rate convergence as new cells are activated and as demands (of User 4 and 7 of BS1) change.

Fig. 11. A heterogeneous network.
\[ p_j^{t+1} = p_j^t + \mu_j^t \mathcal{W}_j(p_j^t, p_{-j}^t), \quad p_{-j}^t = [p_{j1}^t, \ldots, p_{jN}^t] \text{ for all } j < N, \quad (11) \]

where \( \{p_j^t\} \) is a random process that can, for example, represent some form of estimation error while observing the observables \( \{\mathcal{W}_j\} \) that depend upon the parameter \( p \). These algorithms can be approximated by the solution of the ODE (see for e.g. [1]):

\[ \dot{p}_j = \mathcal{W}_j(p) \quad \text{with} \quad \mathcal{W}_j(p) := \mathbb{E}[\mathcal{W}_j(p, V_j)] \quad \text{for all } j \]

and are shown to converge to a zero of the average function \([w_1, \ldots, w_N]\), under appropriate conditions whenever \( \mu_j^t = \mu_t^0 \) for all \( j \).

There are situations in which different agents can update at different rates, i.e., \( \mu_j^t \) need not be the same for all \( j \). For example one might have in (11) for some \( N_1 < N \)

\[ \mu_j^t = \mu_t^1 \quad \text{for all } j < N_1 \text{ and } \mu_j^t = \mu_t^2 \quad \text{for all } j > N_1 \text{ with } \mu_t^1 \neq \mu_t^2. \]

Such situations are analyzed via two time scale based stochastic approximation results (see for example [4,1]) when say \( \mu_t^1 = 0(\mu_t^2) \). These analysis primarily assume that, for any given fixed value of the slower component \( \{p_j^t\} \), the ODE corresponding to the faster components \( \{p_j^t\} \) is

\[ \dot{P}_j(t) = \mathcal{W}_j(P_j(t), P_{-j}) \quad \text{with} \quad \mathcal{W}_j := [w_1, \ldots, w_N] \quad (13) \]

has a uniquely globally stable attractor \( A(P_j) \). Under some more assumptions, it is shown that the trajectory of the slower components is approximated by the ODE (see [1, Theorem 6.1, pp 287], [4, Theorem 1.1])

\[ \dot{P}_j(t) = \mathcal{W}_j(P_j(t), A'(P_j(t))) \quad \text{with} \quad \mathcal{W}_j := [w_{N+1}, \ldots, w_N] \quad (14) \]

and the slower component converges towards the limit set of the above ODE.

So we have two sets of ODEs, joint ODE (12) for single time scale algorithms and two coupled ODEs (13) and (14) for two time scale algorithms. And the comparison of the limits of any algorithm (e.g. UPAMCN), with or without the same update rate by all the agents, can be done by comparing the zeros of the right hand sides of these ODEs. One can notice\(^7\) that the two sets of limit points will be “same” in the following sense:

(a) if \( P_j \) is an attractor (hence a zero of the RHS) of the ODE (14) then \( \{P_j, A'(P_j)\} \) is an attractor of the joint ODE (12);

(b) if \( \{P'_j, P''_j\} \) is an attractor of the joint ODE (12) then necessarily \( P'_j = A'(P''_j) \) (because ODE (13) has a unique attractor for any given \( P_j \) and hence \( P_j \) is an attractor of the ODE (14)).

Remark 1. This implies that the algorithm converges to the same set of limit points, irrespective of the disparities in the update rates. The algorithm (10), that we would like to study, is different from the two time scale algorithms discussed just above. Here, we need the result when say \( \mu_j^t > 0 \) for all \( t \) and \( \mu_j^t > 0 \) only when \( t = k \) for some integer \( k \). We approximate this algorithm in an algorithm in which the slower component is also updated every time slot but with a smaller update co-efficient, that is \( \mu_j^t = \mu_j^t/\kappa \) in the limit \( \kappa \to \infty \). This system can now be analyzed using the two time scale ODEs (13) and (14).

Example Scenario, Low SNR regime: We consider Low SNR regime and a single cell updating fast in comparison with the rest, i.e., as in Eq. (10) with \( N_1 = 1 \). Using [1, Theorem 6.1, pp 287] we show that the UPAMCN converges to the same demand satisfying power profile, as it would have done in case all the users were updating at the same rate (i.e., when \( \kappa = 1 \)).

We consider system C-1-ZF under low SNR regime for which (if \( x \) is small, \( \log(1 + x) \approx x \))

\[ p_{kj}^{t+1} = \frac{1}{\kappa} \frac{p_{kj}^t}{N} \frac{\left( \frac{\mathcal{H}(p_{kj}^t, p_{kj}^t)}{\mathcal{H}(p_{kj}^t, p_{kj}^t)} \right)^q}{1 + \frac{\mathcal{H}(p_{kj}^t, p_{kj}^t)}{\mathcal{H}(p_{kj}^t, p_{kj}^t)}} + \sigma^2_{kj}. \quad (15) \]

Let us say without loss of generality that the BS 1 updates fast, i.e., it updates its power profile every time slot, while the rest update once in \( \kappa \) time slots where \( \kappa \) is very large. We approximate it with a system in which the rest of the components are updated every time slot, albeit with a smaller coefficient \( \mu_t^j/\kappa \) with \( \kappa \to \infty \), say as below (for all \( k, j \)):

\[ p_{kj}^{t+1} = \frac{1}{\kappa} \frac{p_{kj}^t}{N} \frac{\left( \frac{\mathcal{H}(p_{kj}^t, p_{kj}^t)}{\mathcal{H}(p_{kj}^t, p_{kj}^t)} \right)^q}{1 + \frac{\mathcal{H}(p_{kj}^t, p_{kj}^t)}{\mathcal{H}(p_{kj}^t, p_{kj}^t)}} + \sigma^2_{kj} \quad (16) \]

We study the above heterogenous UPAMCN using the two coupled ODEs (13) and (14). The fast ODE (13) for UPAMCN, at slower components \( P_{-1} \), equals:

\[ \dot{p}_{kj} = r_{kj} - c_{kj}(P_{-1}) \quad \text{with} \quad c_{kj}(P_{-1}) := \frac{1}{\sum_{l=1}^N \rho_l} \left( \frac{1}{\sum_{l=1}^N \rho_l} \right) \]

while the ODE (14) corresponding to the slower components (i.e., BS \( j \) with \( j > 1 \))

\[ \dot{p}_{kj} = r_{kj} - p_{kj} h_{kj}(P_{-1}) + z_{kj} \quad \text{for any } k \text{ and } j > 1 \text{ and where} \]

\[ \psi_{kj}(P_{-1}) := z_{kj}(\mathbf{p}_{kj}(P_{-1}), P_{-1}), \quad \mathbf{p}_{kj}(P_{-1}) := [p_{kj}(P_{-1}), \ldots, p_{kj}(P_{-1})] \quad \text{and with} \]

\[ p_{kj}^{t+1}(P_{-1}) = p_{kj}(P_{-1}), \quad \text{and with} \quad p_{kj}^{t+1} := \frac{r_{kj}}{z_{kj}(P_{-1})}. \quad (18) \]

Note in the above that \( \mathbf{p}_{kj}(P_{-1}) \) represents the attractor of the faster ODE when the slower components are fixed at \( P_{-1} \). We obtain the following (Proof in Appendix C):

Theorem 3. Under the assumptions A.1–4 and for the system (16), whenever the interference level \( \gamma_{max} = \max \gamma_l \) is within a limit, the below result is true. For every \( \delta > 0 \), the fraction of time of the tail of the slower components of the UPAMCN algorithm (\( \{P_j^t\}_{t=1}^{\infty} \) with initialization \( P^0 \approx P \)) spends in the \( \delta \)-neighborhood of the limit set, \( L_{slow} \), of the slower ODE (18) tends to one (in probability) as \( t \to \infty \). Further, when \( L_{slow} \) the set of the demand satisfying power profiles is inside the capacity region \( C_{slow}(P) \), it is a part of the limit set \( L_{slow} \) in the sense:

\[ L_{slow} \in \{[\mathbf{p}(P_{-1}), P_{-1}] ; P_{-1} \in L_{slow} \}. \quad \Diamond \]

The above theorem also characterizes the limit set of the ODEs. We show that all the demand satisfying power profiles are indeed the attractors (for this low SNR example) and hence constitute the limit set (see the proof in Appendix C). In fact, the result about the limit set is correct even when we consider all agents updating at the same rate. So the main conclusion is that the UPAMCN is not effected by the variation of the update rates at different agents. The example considered in this section is a restrictive example, and we have partial theoretical justification in this case. However, in the next sub section, via some numerical examples, we indeed show that UPAMCN is unaffected by disparities in the update rates for many general cases.

---

\(^7\) The precise analysis would require some extra conditions and here we are just pointing out the general outline. We obtain the exact analysis for an example in the coming subsection.
6.2. Numerical examples

We consider a network with 3 base stations as in Fig. 11. This network has a macro BS (BS1) and two small cell base stations (BS2 and BS3), both of which update at a faster rate in comparison with the macro BS. Each of them support 8 users. Further, we assume system configuration S2, described in Table 4. Note that S2 is a system which supports only finite number of transmission rates and hence is a more practical example in comparison with the one studied using the two time scale analysis in the previous subsection. To keep it simple, we assume that the demand rates of users in each cell is the same and is given by \( r_i = \frac{1}{12} \) for \( i = 1, 2, 3 \). The interference factor \( \gamma = 0.5 \). We show via this case study the working of the algorithm and claim that due to the inherent nature of the algorithm, UPAMCN can converge and track in a variety of scenarios.

The power profile at the macro cell is updated once in \( \kappa \) time slots while the small cells update every time slot. We plot the UPAMCN power updates for two values of \( \kappa \) (1 and 10) in Figs. 12 and 13. We see from the figures that the demands are met, power profile converges to the same point irrespective of \( \kappa \), albeit with different speeds. We further observe from the two figures that the convergence speed of the slower components is proportional to \( \kappa \), which is quite intuitive (the effective number of updates before convergence remain the same). More surprisingly the convergence speed of the faster components does not depend much upon \( \kappa \) (though both the systems are coupled). The reason for this being the following: the attractors of the faster components \( P_i^\kappa(p_i) \) are (Lipschitz) continuous in the slower component \( p_i \) and hence will vary little with small changes in the slower component and hence it appears to have converged faster.

We further illustrate the robustness of UPAMCN algorithm against update rate disparities in Table 6. Here, we tabulate the converged power profile of all the eight users of the macro cell with different \( \kappa \). We do not tabulate the quantities corresponding to the small cells as they do not change much with \( \kappa \). We observe

![Power convergence](image1)

**Fig. 12.** Power convergence with heterogenous users.

![Rate convergence](image2)

**Fig. 13.** Rate convergence with heterogenous users.

<table>
<thead>
<tr>
<th>( \kappa )</th>
<th>Demand satisfying NE (converged power)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.011, 0.023, 0.036, 0.050, 0.065, 0.081, 0.098, 0.1168</td>
</tr>
<tr>
<td>2</td>
<td>0.0121, 0.0242, 0.0373, 0.0513, 0.0664, 0.0824, 0.0998, 0.1183</td>
</tr>
<tr>
<td>5</td>
<td>0.0118, 0.0240, 0.0368, 0.0508, 0.0656, 0.0817, 0.0987, 0.1170</td>
</tr>
<tr>
<td>10</td>
<td>0.0120, 0.0241, 0.0371, 0.0510, 0.0660, 0.0821, 0.0993, 0.1178</td>
</tr>
<tr>
<td>25</td>
<td>0.0120, 0.0242, 0.0375, 0.0515, 0.0666, 0.0828, 0.1001, 0.1186</td>
</tr>
<tr>
<td>50</td>
<td>0.0120, 0.0242, 0.0374, 0.0514, 0.0665, 0.0827, 0.0999, 0.1184</td>
</tr>
<tr>
<td>100</td>
<td>0.0120, 0.0243, 0.0375, 0.0516, 0.0667, 0.0829, 0.1002, 0.1189</td>
</tr>
</tbody>
</table>

System S2 with heterogenous users convergence with different \( \kappa \)
that UPAMCN converges to the same power profile for all the values of $\kappa$. In Table 7, we also tabulate the number of time steps needed to converge to a ball which is within 5 % of the demand rates for all the users of the macro cell. We see that (roughly) the number of time steps for the convergence is proportional to $\kappa$.

### 7. Conclusions

Mobile broadband users demand certain rates depending on the end application and QoS requirements. The base station serving these users has to allocate power to satisfy user demands operating within its own total power budget. Intra-cell and inter-cell interference diminish the available rates in multicell networks. Neighboring base stations can co-operate to exchange some form of channel state information depending on backhaul capacity and processing power to alleviate interference and thus enhance achievable rates. Further, system specific components like modulation, coding, rate allocation, channel estimation and synchronization impact the achievable rates and hence the power allocation. We observe that the power profile satisfying the demands depends upon the exact configuration of the system and we propose an universal power allocation algorithm which works in a multitude of such settings. The stochastic approximation based universal power allocation algorithm runs at each BS, independently and simultaneously to meet the user demands as long as the demands are achievable. The power allocation is formulated as a game problem. A system specific capacity region is defined and the proposed algorithm is analyzed using an ODE framework. The proposed algorithm works well in a multitude of system configurations as demonstrated via simulations and analysis: it converges to the system specific power profile, which satisfies the demands asymptotically.

The proposed algorithm assumes that the serving BS always has sufficient amount of data to transmit. However, in many applications, the data is available in real time. We mention a possible extension of the same in the paper.

Cellular networks with heterogeneous users can have different processing powers and may update their power profiles at different rates. We show that the proposed algorithm works irrespective of the disparities in the update rates using two time scale analysis and numerical simulations: the algorithm converges to the same power profile as long as the demands and the system configuration remain the same, irrespective of the disparities in update rates.

### Acknowledgments

This research is carried out in the framework of the INRIA and Alcatel-Lucent Bell Labs Joint Research Lab on Self Organized Networks and the Alcatel-Lucent chair on Flexible Radio. The work of V. Kavitha is supported by CEFIPRA.

### Appendix A. Example systems

1. **Asymptotic Ideal Rate system:** In a multicellular system with large number of antennas at the BS and large number of users, the rate for a given CS can be obtained using random matrix theory. For example, in [11] the asymptotic rates are derived for a zero forcing (ZF) precoder. It is shown that for almost all realizations of CS, the rate can be approximated by the expression given below in Eq. (19). Further, we consider a system in which, the base stations use channel coding schemes to transmit very close to the theoretical rates. When this system (which we call as asym-ideal-zero forcing or in short A-I-ZF according to our notations) uses power profile $P$ and when the channel state (CS) is $\mathcal{H}$, the BS $j$ transmits to the user $k$ at ([11]) $(M > K)$:

$$R_{kj}^{A-I-ZF}(P, \mathcal{H}) \approx \log \left(1 + \frac{P_{kj}}{\sum_{l \neq j} (\frac{1}{2} \mathbf{w}_l^H \mathbf{w}_l + \sigma_{kj})} \right)$$

(19)

where, $\beta = M/K$ is the ratio of number of transmit antennas on the BS to the number of users and $\gamma_l \in (0, 1)$ represents the interference from cell $l$. This rate is same for almost all CS $\mathcal{H}$ as it is an asymptotic rate. Similar expression is available for the case with $M = K$ in [11].

2. **Ideal rates using complete CSIT:** If the number of antennas/number of users is not large enough, the asymptotic results are not accurate. If BS has access to CSIT (and if each BS could channel code to obtain rates closer to the ideal rate) then with ZF precoder it transmits at rate:

$$R_{kj}^{A-I-ZF}(P, \mathcal{H}) = \log \left(1 + \frac{P_{kj}}{\sum_{l \neq j} (\frac{1}{2} \mathbf{w}_l^H \mathbf{w}_l + \sigma_{kj})} \right)$$

$$Q_l := \mathbf{H}_l^H \left( \mathbf{H}_l^H \mathbf{H}_l - \sigma_{kl} \right)^{-1} \mathbf{H}_l^H$$

For the same configuration, but without transmitter precoding, the instantaneous transmission rate (as obtained using Shannon’s capacity expression), from Eq. (2) is:

$$R_{kj}^{C-I-ZF}(P, \mathcal{H}) = \log \left[1 + \frac{P_{kj}}{\mathbf{H}_l^H \left( \mathbf{H}_l^H \mathbf{H}_l - \sigma_{kl} \right)^{-1} \mathbf{H}_l^H} \right]$$

(20)

3. **Finite number of Rates:** Ideal rate systems are not realistic, they can’t be implemented in practice. We consider a system, in which the BS can transmit at one of the available discrete rates from the set $\mathcal{R}$. When transmitter has CSIT, it knows the exact theoretical rate and hence will pick the largest rate from set $\mathcal{R}$ that is smaller than the current theoretical rate:

$$R_{kj}^{C-I-ZF}(P, \mathcal{H}) = \inf_{r \in \mathcal{R}} \left[ r \leq R_{kj}^{C-I-ZF}(P, \mathcal{H}) \right]$$

(21)

$$R_{kj}^{C-I-ZF}(P, \mathcal{H}) = \inf_{r \in \mathcal{R}} \left[ r \leq R_{kj}^{C-I-ZF}(P, \mathcal{H}) \right]$$

(22)

4. **Rate adaptation Without CSIT:** It is once again not realistic to assume the knowledge of complete CSIT. There are many schemes that estimate the rate blindly or using some partial CSIT (e.g. [12]). The UPAMCN algorithm is a very general algorithm and works with all those systems which satisfy assumptions A.1–4. These are quite simple assumptions and most of the systems can satisfy these and hence the algorithm works for majority of the blind/partial CSIT systems.

We explain one such blind system wherein, the BS estimates the transmission rates without knowledge of CSIT. Each time, the BS begins by attempting at the highest available rate $r_1$. If the data is not received correctly (information

### Table 7

<table>
<thead>
<tr>
<th>$\kappa$</th>
<th>User 8 Small cell (BS 2)</th>
<th>User 8 Macro cell (BS1)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>506</td>
<td>507</td>
</tr>
<tr>
<td>2</td>
<td>505</td>
<td>957</td>
</tr>
<tr>
<td>5</td>
<td>501</td>
<td>2321</td>
</tr>
<tr>
<td>10</td>
<td>506</td>
<td>4540</td>
</tr>
<tr>
<td>25</td>
<td>507</td>
<td>11269</td>
</tr>
<tr>
<td>50</td>
<td>506</td>
<td>22907</td>
</tr>
<tr>
<td>100</td>
<td>506</td>
<td>45014</td>
</tr>
</tbody>
</table>
obtained via a feedback from the receiver), the BS sends some more information about the same data packet so that the overall rate now is the second highest \(r_2\). This procedure repeats until the two agree upon the correct rate. We assume that this rate adaptation system is always successful, i.e. it can estimate the actual rates without errors. Such a system does not require CSIT, however the final rate at which the transmission takes place depends upon the current channel state in exactly the same way as in the case of C-D (or A-D for large antenna and user case and note there is no channel coding in this case as there is no CSIT) and hence,

\[
E_{k,j}^{N, RA, NO}(P, \mathcal{H}) = R_{k,j}^{C, D, NO}(P, \mathcal{H})
\]  

(23)

(5) Rate Adaptation with local CSIT: All the base stations have local CSIT, i.e., BS \(j\) knows the \(H_j\) part of CS. However they can’t estimate the current rates just based on local CSIT. So, they once again use rate adaptation technique as in the system (4). They can however design a better system by using for example a zero forcing precoder. In this case, as in system (4) the rate will be adapted to the actual underlying rate and hence will be same as that in C-D-ZF:

\[
R_{k,j}^{l, RA, ZF}(P, \mathcal{H}) = R_{k,j}^{C, D, ZF}(P, \mathcal{H})
\]  

(24)

(6) Rate Adaptation with errors: There can be some errors in rate adaptation algorithm of system (4) or (5). In this case

\[
E_{k,j}^{N, RA, NO}(P, \mathcal{H}) = R_{k,j}^{N, RA, NO}(P, \mathcal{H}) - E_{k,j}^{N, RA, NO}(P, \mathcal{H})
\]  

(25)

\[
R_{k,j}^{l, RA, ZF}(P, \mathcal{H}) = R_{k,j}^{l, RA, ZF}(P, \mathcal{H}) - E_{k,j}^{l, RA, ZF}(P, \mathcal{H})
\]  

(26)

where (assuming independent errors) \(E_{k,j}(\bar{r})\) can take values in the subset \(R \cap \{r \leq \bar{r}\}\) with a given probability distribution.

Appendix B. Proofs

Proof of Theorem 1. As a first step, we rewrite the algorithm as in [1]:

\[
Y_{t,k,j} := r_{k,j} - d_{k,j}^{t,k,j}, \quad p_{k,j}^{t+1} = \Pi_{\Gamma_{t,k,j}}[p_{k,j}^{t} + \mu t Y_{t,k,j}].
\]

Define, \(\mathcal{F}_t := \sigma\left(\{Y_{t,k,j}^{t+1}, k,j\}_{t=1}^{\infty}\right)\), \(\mathcal{F}_t\) for all \(t \leq T\) and let \(\xi_t\) represent the expectation w.r.t. \(\mathcal{F}_t\), the filtration. Under the assumptions A2 and A3 clearly the condition expectation

\[
E_t[Y_{t,k,j}] = g_{k,j}(Y_{t,k,j}) := r_{k,j} - R_{k,j}(P_{k,j})
\]

for all \(k, j \) and \(t\). For every \(j\), the constraint set \(A_j\) satisfies the assumption (A3.2), page 107 of [1]. By assumption A3 \(\{Y_{t,k,j}\}_{t}^{\infty}\) is uniformly integrable and hence satisfies assumption A2.1, pp. 258 [1]. They also satisfy the assumptions A2.3 to A2.7 of pages 258, 259 [1] with \(g_t = g = g_{k,j}\) and with \(\beta_t = 0, \xi_t = 0\) for all time. Assumption A2.2, pp 258, [1] is satisfied because of our Assumption A4. Let \(z_j\) represent the projection or constraint term, the minimum force needed to keep the vector \(p\) in \(A_j\). Then by Theorem 2.3, pp. 259, [1] the UPAMCN algorithm trajectory \(p_{n}\) converges weakly to the trajectory of the solution of the ODE (8) (in the sense as explained in [1]). Further by the same theorem of [1], for any \(\delta > 0\), the fraction of time that the tail sequence \(\{P^t\}_{t=\infty}\), with initializations \(p_{k,j}^{t}\) for every \((k,j)\), spends in the \(\delta\)-neighborhood of the limit of set of the above ODE (8), \(B_{\delta(t,\infty)}\), goes to one (in probability) as \(t \to \infty\). \(\diamondsuit\)

Proof of Lemma 2. The boundedness assumption A3 is direct for discrete rate systems and is also true for ideal rate systems as seen from the formulas. The ideal rates are point wise continuous and are bounded and hence by bounded convergence theorem satisfy the continuous assumption A4. The same for the discrete rates is given by Lemma 3.

The continuity assumption A4 now also holds for the rate adaptation system with errors, L-RAE-NO, whenever the statistics of the errors \(\{E_{k,j}(\bar{r})\}\) are independent of the power profile or when they are continuous in \(P\). Thus for all the systems considered in this paper Theorem 1 applies.

Conditions for existence of demand meeting NE: For all the systems considered so far, the hypothesis of Lemma 1 is satisfied, i.e., \(\Pi_{NO}\) is non empty whenever the power constraints are sufficient to cater to the demand rates. This fact is established by the continuity of the average rates w.r.t. the power profile, i.e., the establishment of the assumption A4. To be precise Lemma 1 is satisfied, i.e., \(\Pi_{NO}\) is non empty whenever for all \(i, j, r_{ij} < \sup P \in \Pi_{k,j}\).

Lemma 3. The average rates \(R_{k,j}(P_{k,j})\) for systems C-D-ZF and C-D-NO are continuous w.r.t. power profile \(P\) for all \(i, j\).

Proof. Let \(R_{k,j}(P, \mathcal{H})\) represent the corresponding ideal rate (the rate before discretization) for the given CS \(\mathcal{H}\). From all the rate formulas in this paper, we can see that these rates bounded and are continuous in \(P\), for all \(\mathcal{H}\). For the discretized systems, the average rates can be written as,

\[
R_{avg,k,j}(P) = \sum_{i \in N} q(i, P) r_{ij}, \quad \text{where}
\]

\[
q(i, P) := \int_{\{s \in R \cap \{r \leq \bar{r}\}\} \cap \{s \in R \cap \{r \leq \bar{r}\}\} \cap \{s \in R \cap \{r \leq \bar{r}\}\}} d\Gamma'(\mathcal{H})
\]

with \(d\Gamma'\) representing the Gaussian measure. For a given \(P\), the probability of the sets of the type (the boundaries of the sets used while defining the indicators in (27))

\[
\Gamma'\{\mathcal{H} : R_{k,j}(P, \mathcal{H}) = r_{ij}\} = 0,
\]

because of the continuity of the Gaussian measure. Hence, the point wise functions of integral (27) are continuous w.r.t. \(P\) for almost all \(\mathcal{H}\). Thus the lemma follows by bounded convergence theorem. \(\diamondsuit\)

Appendix C. Proofs related to two time scale algorithms

Proof of Theorem 3. We obtain this proof via the [1, Theorem 6.1, pp 287]. The [1, Assumptions A6.0, A6.1, A6.2, A6.3 and A63.5, pp 287] hold for this example as is shown in the proof of Theorem 1. Thus it remains to prove the [1, Assumption A6.4], in order to apply [1, Theorem 6.1, pp 287]. The ODE corresponding to the faster component (BS 1) for a given value of the other user’s parameters \(\mathcal{H}_1\) is given by (17) (see [1] for details). It is easy to see that this ODE has unique solution,

\[
p_{k,1}(t) = p_{k,1}(P_{-1}) - e^{-x_{k,1}(P_{-1})t} \text{ with}
\]

\[
p_{k,1}(P_{-1}) = \frac{r_{k,1}}{x_{k,1}(P_{-1})} \quad \text{for all } k
\]

and that \(p_{k,1}(P_{-1})\) is its unique globally stable attractor. Further, there exists a constant \(\nu < \infty\) such that,
\[ |x_k(P_{-1}) - x_k(P_{-1})| \leq \frac{V}{\gamma_{\text{max}}^3} |P_{-1} - P_{-1}| \]

with

\[ \gamma_{\text{max}} := \max_{l,k} \gamma_l \]

for all \( k \),

and thus the function \( p_l \) is locally Lipschitz, satisfying [1, Assumption A6.4]. By [1, Theorem 6.1, pp. 287], the tail of the trajectory of UPAMCN spends its time mainly in the limit set of the mean ODE (see [1] and with \( \psi_{l}(P_{-1}) := x_k \left( \left[ p_l(P_{-1}), P_{-1} \right] \right) \),

\[ r_{kj} = r_{kj} - p_{kj} \psi_{l}(P_{-1}) + z_{kj} \]

for any \( k \) and \( j > 1 \).

We now characterize this limit set, denoted by \( L_{\text{slow}} \). We will show that every internal (which is not on the boundary of the constraint set) demand satisfying power profile is a part of \( L_{\text{slow}} \); i) it is easy to see that \( [p_l(P_{-1}), P_{-1}] \) is an internal demand satisfying power profile if and only if \( P_{-1} \) is an internal zero of the RHS of the above ODE; ii) below, we will show that every internal zero of the RHS of the above mean ODE, will be an asymptotically stable attractor and hence is in \( L_{\text{slow}} \).

Let \( P_{-1} \) be any internal zero of this ODE and let \( e_{kj} := r_{kj} - P_{-1} \) for every \( k \) and \( j > 1 \). Consider a neighborhood of \( P_{-1} \) which is contained inside the constraint set (hence the projection term would be zero) and in this neighborhood we have (note \( p_{kj} \psi_{l}(P_{-1}) = r_{kj} \))

\[ e_{kj} = p_{kj} \psi_{l}(P_{-1}) - p_{kj} \psi_{l}(P_{-1}) \]

\[ = -e_{kj} \psi_{l}(P_{-1}) + p_{kj} \psi_{l}(P_{-1}) - \psi_{l}(P_{-1}) \cdot \]

Let \( E \) represent the vector \( \{ e_{kj} \}_{kj>1} \). There exists a constant \( c \) depending upon the radius \( r \) and \( P_{-1} \) such that for all \( E \) with \( |E| \leq r \) (using the upper bound (28)),

\[ E, E > \sum_{kj>1} (-\psi_{l}(P_{-1}) + cV_{\text{max}}) |E|^2. \]

Thus by [14, Global existence theorem, pp 169-170],

\[ |E(t)| \leq e^{\sum_{kj>1} (-\psi_{l}(P_{-1}) + cV_{\text{max}}) |E|^2} \]

when initial condition \( E(0) \in \{ E : |E| < r \} \).

Hence as the interference reduces, i.e., as \( V_{\text{max}} \to 0 \) the exponent term becomes negative (in which case, the error tends to zero asymptotically). For these small values of interference, every internal zero is an asymptotically stable attractor. \( \diamond \)

References


