MIMO RADAR MODELING
THROUGH RANDOM VANDERMONDE MATRICES

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ABSTRACT
A MIMO radar system is conveniently modeled via random matrices, and its optimal design strongly relies on spectral properties of the matrix exploited to build the model itself. We offer a way to model a High Resolution Radar (HRR) detection in the MIMO case, based on recent results on the asymptotic spectral analysis of random Vandermonde matrices with entries lying on the unit circle. Achievability of compact-form expressions for the design parameters sought in the multiple-transmitter-multiple-receiver case is investigated, and together with the results of such a starting analysis, some open mathematical questions that arise from the new model formulation are listed and discussed.

1. INTRODUCTION
The seminal idea of Woodward and Davies of applying information theory to radar systems analysis [1, 2] has been only rarely exploited in radar waveform design [3], [4], and references therein; more precisely, joint information- and estimation-theoretic criteria are adopted in several works by the same authors (see, e.g. [4, 5]) to optimally shape the transmitted waveform from a multiple-antenna equipped radar transmitter which aims at detecting an extended target at high resolution. The receiver too is assumed to be equipped with multiple antennas, and the model applies to both mono- or bi-static scenario.

The main finding of the abovementioned works is the dependence of the parameters of interest in the waveform design strategy on the spectrum of the Gram matrix associated to the (random) matrix modeling the input-output relationship between the transmitted waveform from each sensor and the target echo at any receive antenna. For sake of simplicity, we will refer hereinafter to the input-output matrix as the channel matrix as in any linear framework for MIMO wireless communication analysis.

Stationarity assumptions in [4] yields to a Toeplitz [6] structure for the blocks of the channel matrix modeling the signal reflection on the target from a single transmit antenna to a single receive antenna. This feature of the channel matrix, when taking into account also the unknown reflection angle from each of the elementary scatterers, allows, as noticed in [7], to model the Gram matrix associated to the channel matrix as random Vandermonde. Spectral properties of such kind of matrices have been only recently investigated in some works [7, 9, 10] under different assumptions on the entries distribution (especially on their dependence) and on the matrix aspect-ratio convergence properties.

While the works appeared on the line of [4] have shed light on the connection between large system properties and waveform design, we aim with this contribution at investigating the performance-side impact of the Vandermonde modeling for MIMO radar, as well as at stressing some open problems in Vandermonde matrices spectrum characterization arising from the MIMO detection setting itself and that have not yet been encountered when dealing with random Vandermonde determinants in other applications like finance, cognitive radio, security and, in general, wireless communications issues (for a list of references on each subject, please refer to [9], where a comprehensive analysis of the Vandermonde modeling state-of-the-art is provided).

The paper is structured as follows: Section 2, contains some essential background material on (large) random Vandermonde matrices spectral properties. Section 3 reports the model for a MIMO HRR system in terms of Vandermonde matrices, while Section 4 discusses the available results on the MIMO HRR system matrices and open issues. Conclusions are given in Section 5.

2. MATHEMATICAL BACKGROUND
This section addresses some essential definitions in random matrix theory that will be useful in the following, as well as the exploited notation. Throughout the paper, matrices are denoted by uppercase boldface letters, vectors by lowercase boldface; \( \mathbb{E}[\cdot] \) denotes the statistical expectation, \( {(\cdot)}^{\dagger} \) indicates the conjugate transpose operator, \( \|\cdot\| \) and \( \text{Tr}(\cdot) \), respectively, the determinant and the trace of a square matrix, and \( \|\cdot\|_2 \) for the euclidean norm.

Definition 2.1 Let us consider an \( N \times N \) Hermitian matrix \( \mathbf{A} \). The averaged empirical cumulative distribution function of the eigenvalues (also referred to as the averaged empirical spectral distribution (ESD)) of \( \mathbf{A} \) is defined as

\[
F^N_X(\lambda) = \frac{1}{N} \sum_{i=1}^{N} \mathbb{E}[1\{\lambda_i(\mathbf{A}) \leq \lambda\}],
\]
where \( \lambda_1(\mathbf{A}), \ldots, \lambda_N(\mathbf{A}) \) are the eigenvalues of \( \mathbf{A} \) and \( 1\{\cdot\} \) is the indicator function. If \( F^N_X(\lambda) \) converges as \( N \to \infty \), then

\[2\] It is worth to note that in [9, Appendix F] a useful connection between Vandermonde moments evaluation and the analysis on random Toeplitz carried on in [8] has been pointed out too.
the corresponding limit (asymptotic ESD, AESD) is denoted by $F_A(\cdot)$. The corresponding asymptotic probability density function is denoted by $f_A(\cdot)$.

**Definition 2.2** The aspect-ratio of a $N \times L$ random matrix is the number $\beta = \lim_{K,L \to \infty} \frac{k}{L}$, provided the limit is finite.

**Definition 2.3** [11] The $\eta$-transform of the random matrix $A$ is defined as

$$\eta_A(\gamma) = \mathbb{E} \left[ \frac{1}{1 + \gamma \lambda} \right]$$

(1)

where $\gamma$ is a scalar and $\lambda$ is a random variable distributed as the asymptotic eigenvalues of $A$.

By denoting with $\mathbb{E}[\lambda^p]$ the $p$-th asymptotic moment of $A$, $\eta_A(\gamma)$ can be regarded as a generating function for the asymptotic moments of $A$, i.e. [11].

$$\eta_A(\gamma) = \sum_{p=0}^{\infty} (-\gamma)^p \mathbb{E}[\lambda^p],$$

(2)

whenever the moments of $A$ exist and the series in (2) converges.

The $\eta$-transform, introduced in [11], is intimately related to other integral transforms defined over the spectrum of a random matrix. For a detailed discussion on the links between such transforms and the Fourier transforms usually exploited in free probability, the reader is referred to [11, Ch. 2]. Recently, in [13] some more general results appeared on the connection between the $\eta$-transform and other functions of the asymptotic moments of a random matrix under quite generic assumptions.

The random matrix we will mainly concerned with throughout the paper are the Vandermonde ones, that are classified as follows:

**Definition 2.4** We define Vandermonde matrix an $N \times L$ matrix $V$ whose $(k, \ell)$-th entry can be expressed as $x_k^{\ell-1}$ [14].

**Definition 2.5** An $N^d \times L$ matrix $V$, with complex exponential entries lying on the unit circle, is a $d$-fold Vandermonde matrix if its generic entry can be written as

$$V_{V}(k,q) = \frac{1}{\sqrt{N^d}} e^{-2\pi i \theta_j f(x_k)},$$

(3)

where $\ell = [\ell_1, \ldots, \ell_d]^T$ is a vector of integers, $\ell_m = 0, \ldots, N-1$, $m = 1, \ldots, d$, the function $f$ is evaluated component-wise on the random vectors $x_k$, and the function

$$V(\ell) = \sum_{m=1}^{d} N^{m-1} \ell_m,$$

(4)

maps the vector $\ell$ onto a scalar index. According to [9], we will refer to the quantity $2\pi f(X_k)$ as the phase distributions of the Vandermonde matrix at hand. Notice that for a $d$-fold Vandermonde matrix the phase is a vector while for $d = 1$ the phase is a scalar quantity.

Notice that the assumption of the entries lying on the unit circle is crucial to the asymptotic convergence of the spectrum and will be retained through the paper. When $d = 1$, we will refer to (3) as a random Vandermonde matrix. Notice further that, due to the previous definitions, we will have $\beta = \frac{1}{N}$ for a Vandermonde matrix and $\beta = \frac{1}{N\nu}$ for the $d$-fold version.

The application we are interested in leads us to consider the function $f = \sin(\gamma x)$.

### 3. SYSTEM MODEL

HRR is often exploited in military applications for detecting extended targets. To address the problem of detection through HRR, several mathematical tools have been proposed in the literature. One of the simplest, but effective, approaches is to resort to random matrix theory and exploit a linear model involving Vandermonde matrices. Recall that the echo from an extended target which is resolvable in $L$ elementary scatterers can be viewed as the impulse response of a linear filter of length $v$, where $L = v + 1$. Let us assume that the transmitted signal has length $N$. Then, by considering a (mono or bistatic) scenario for HRR detection, the received signal can be written as [4, and refs. therein]

$$r = VP^{1/2}s + n$$

(5)

where $\mathbb{E}[\{P\}] = 1$ and $V$ is a $N \times L$ Vandermonde matrix with generic entry$^3 \{V\}_{n,q} = \exp(-2\pi \sin \theta_j)/\sqrt{N}$. $\theta_j$ is the angle of arrival of the echo reflected on the $q$-th elementary scatterer constituting the target, and $s$ is the target impulse response of length $L$. The diagonal matrix $P$ accounts for the (eventually) different power levels of the echoes coming from each of the $L$ scatterers constituting the overall extended target, which is usually assumed to have a Gaussian vector impulse response [4]. When dealing with the MIMO case, the channel matrix turns out to be a $d$-fold Vandermonde, where $d$ is the number of antennas deployed at the receiver. This way, the generic entry of $V$ will depend, as in the multuser multitenna case with line of sight contribution [7], [9, Sec. VI], on the vector of the arrival angles on each of the $d$ receiving antennas, for each echo reflected from the $q$-th elementary scatterer.

For the linear model in (5), the Mean Square Error (MSE) on the LMMSE estimate of $s$, normalized over the size $L$ of $s$, is obtained as [15]

$$\text{MSE} = \text{tr} \{(\gamma VPV^\dagger + I)^{-1}\}$$

(6)

For large systems, we also define the asymptotic MSE as

$$\text{MSE}_{\infty} = \lim_{M,N \to \infty} \text{MSE} = \mathbb{E} \left[ \frac{1}{1 + \gamma \lambda} \right]$$

(7)

where $\lambda$ is a random variable distributed as the asymptotic eigenvalues of $VPV^\dagger$ [10].

From (1) and (7), we can note that the expression of $\text{MSE}_{\infty}$ can be written through the $\eta$-transform as

$$\text{MSE}_{\infty} = \eta_{HHH}(\gamma)$$

(8)

In the following, particular emphasis will be on the asymptotic moments, from which the $\eta$-transform evaluation can be carried out.

$^3$In this case, $n = 0, \ldots, N - 1$, and $q = 0, \ldots, v$
4. MODEL ANALYSIS

Mirroring the MMSE design criterion of [4], we aim at providing compact expressions for the MSE in the target impulse response parameter estimation, based on recent findings on large random Vandermonde matrices spectra. Throughout the Section, \( \mathbb{E}[\lambda_k^p] \) will denote the \( p \)-th moment of a \( d \)-fold random matrix; as a consequence, we will denote as \( \mathbb{E}[\lambda_k^p] \) the \( p \)-th asymptotic moment.

4.1 Performance discussion

Notice first that, if we were detecting the target over a very narrow angle spread, \( \sin(\theta_i) \approx \theta_i \), then, asymptotically in the number of the receive antennas, results in \((10, 7)\) apply, respectively, for the case of \( \mathbf{P} = \mathbf{I} \) and \( \mathbf{P} \neq \mathbf{I} \). Specifically, when \( d \) grows large, the \( p \)-th asymptotic moments of \( \mathbf{VV}^\dagger \) coincides with that of the Marcenko-Pastur law \([10, \text{Lemma 6.1}]\) and is given by

\[
    \mathbb{E}[\lambda_k^p] = \sum_{k=1}^{p} T(p, k) \beta^{p-k} \tag{9}
\]

while those of \( \mathbf{VPV} \) with \( \mathbf{P} \neq \mathbf{I} \) are given by the well-known Bai-Yin formula \([11, \text{and references therein}]\), namely\footnote{Herein, following \([11]\), we suppose a vector of \( k \) integers \( p_1, \ldots, p_k \) is partitioned into \( n \) equivalence classes under the equivalence relation \( a \equiv b \), and the cardinalities of the equivalence classes are given by \( f_1, \ldots, f_n \), then we can define the following function:

\[
    f(p_1, \ldots, p_k) \triangleq p_1! \cdots p_k!.
\]

For example, \( f(1, 1, 4, 2, 1, 2) = 3! \cdot 2! \cdot 1! \).}

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